A Hopscotch Method for the Korteweg–de-Vries Equation

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A Hopscotch algorithm is derived for the Korteweg–de-Vries equation. The method is analyzed with respect to stability and dispersion. The scheme is shown to be conservative and to possess a minimal phase error. Numerical results are reported for a single soliton solution and the interaction of two solitons with different velocities is investigated.

1. INTRODUCTION

The Korteweg–de-Vries equation (KdV) introduced in [6] describes the behavior of one-dimensional shallow water waves with small but finite amplitude. More recently, this equation also has been found to describe wave phenomena in anharmonic crystals, bubble liquid mixtures, and plasma physics; details may be found in Gardner and Morikawa [2], Washimi and Taniuti [15], Kruskal [7], Zabusky [18], and van Wijngaarden [16, 17].

Theoretical aspects of solutions of the KdV equation recently have attracted attention. In particular, the problems of existence and uniqueness for solutions for certain classes of initial functions have been studied; see, for example, Lax [8] and Sjoberg [13]. Also, Lax [9] and Gardner *et al.* [1] have examined the existence of *solitary wave* or *soliton* solutions.

The physical models described by the KdV equation represent situations requiring large-scale time calculations. Consequently, any numerical method proposed for calculating the solution of the mathematical equation must possess at least two properties. First, the method must represent faithfully amplitudes of the solution for many time steps in the calculation. Consequently, a method with damping properties as time increases, must reduce the size of the solution and thus be inappropriate. (This type of equation is in strict contrast to the nonlinear hyperbolic systems in which shocks occur and dissipative mechanisms are required to produce a narrow wave front). Second, since the position of wave fronts is as important as the amplitude of these waves, the proposed method must be capable

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of predicting such wave fronts with minimal error. Hence, the phase error of the method must be small since, over a long time calculation, large phase errors can produce solutions that are completely out of phase with the (unknown) theoretical solution. (i.e., the method can produce solutions whose amplitude is exact, but whose phase is in error by $\pi/2$; a meaningless solution would then be obtained).

The first property of such methods is simple to achieve. The method should be conservative (i.e., nondissipative; see Richtmyer and Morton [12]). The second property is more difficult to ensure. In an attempt to achieve both these properties, Zabusky and Kruskal [19] proposed a leap frog-like scheme. This is a three-level scheme that is second-order accurate in time. In [19], it is shown that the method is conservative so that the first property described above is satisfied. No attempt is made to analyze the phase properties of their scheme in [19] (however, see Vliegenhart [14].) The numerical results reported appear to justify its choice as an excellent method. However, such a method suffers from two disadvantages. First, being a three-level scheme, the method requires an extra level of storage at each time step of the calculation. For a calculation requiring many gridpoints, this extra storage can be substantial. Second, the three-level scheme requires a starting procedure to produce the extra level of data at $t = 0$. In fact, Vliegenthart [14] describes a two-level scheme to produce the required data at, say, $t = \tau$.

In the present paper, we propose an alternative algorithm to the Zabusky–Kruskal method, which does not suffer from either of the disadvantages described above. This novel algorithm is based on the Hopscotch philosophy described by Gourlay [3]. The method is a two-level scheme that is proved to be conservative in Section 3. The algorithm requires the solution of a constant coefficient tridiagonal system of equations. The work required is minimal, being of $O(n)$ operations since the tridiagonal matrix can be stored in factorized form requiring only a set of back substitutions at each time level to obtain the necessary solution.

For a discretization of time and space into steps τ and h, respectively, it was found that the truncation error of the Hopscotch method is $\tau O(\tau^2 + (\tau/h)^2 + h^2)$, which, under the assumption, necessitated for stability, that $\tau \equiv O(h^3)$ reduce to $\tau O(\tau^2 + h^2)$. This is precisely the form of the truncation error for the Zabusky–Kruskal scheme, ignoring the necessary starting procedure. (In fact, the convergence rate of the Zabusky–Kruskal scheme has not been demonstrated when the starting procedure is of lower accuracy, although in practice the lower order does not seem to detract from the excellence of the numerical results obtained.) The phase error of the novel scheme is analyzed in Section 3, where it is shown that this should be minimal.

The novel algorithm is described in Section 3. Numerical results obtained using the algorithm for a sequence of initial conditions are described in Section 5.

2. BACKGROUND THEORY

Consider the KdV equation

$$u_t + \beta u \cdot u_x + \epsilon u_{xxx} = 0 \tag{1}$$

where β and ϵ are positive constants. We write Eq. (1) as

$$u_t = Lu \tag{2}$$

where $L \equiv -\beta u(\partial/\partial x) - \epsilon(\partial^3/\partial x^3)$. Let x be discretized into steps of length h.

We propose an appropriate discrete approximation to L, say L_h, so that Eq. (2) becomes

$$u_t = L_h u + O(h^\sigma) \qquad \sigma > 0,$$

where the truncation error term $O(h^\sigma)$ arises from the difference between Lu and $L_h u$. Suppose $w(t)$ is a semidiscrete solution, i.e., w satisfies

$$dw/dt = L_h w, \qquad w(0) = u_0 \qquad \text{(given).}$$

If we solve this ordinary differential equation formally, we obtain

$$w(t) = \exp\left[\int_{m\tau}^{(m+1)\tau} L_h(\theta)\,d\theta\right] w(t - \tau), \tag{3}$$

where we have denoted $t \equiv m\tau$, with m a nonnegative integer. To obtain discrete in time approximations we are required to approximate both to the integral in Eq. (3) and to the resulting exponential. If we propose a rectangular rule for the integral in Eq. (3), namely,

$$\int_{m\tau}^{(m+1)\tau} L_h(\theta)\,d\theta \simeq \tau L_h(\theta),$$

we make a truncation error $O(\tau^2)$, for $\theta \neq (m + \frac{1}{2})\tau$. For $\theta = (m + \frac{1}{2})\tau$, the truncation error, corresponding to the resulting midpoint rule, is $O(\tau^3)$. For our present application, $\theta = m\tau$ is adequate. If we now propose a (0, 1) Padé approximation to $\exp \tau L(m\tau)$, we obtain

$$v(t) = [I + \tau L_h(m\tau)]\,v(t - \tau)$$

and the truncation error is $\tau O(\tau + h^\sigma)$, where I is the unit operator.

Introducing the notation $v_m \equiv v(\,,m\tau)$, we have

$$v_{m+1} = [I + \tau L_h(m\tau)]\,v_m . \tag{4}$$

Following Gourlay [3], we may use Eq. (4) as the basis of a class of Hopscotch methods, namely,

$$v_{m+1} + \tau\theta_{m+1}L_h((m+1)\tau)\, v_{m+1} = v_m + \tau\theta_m L_h(m\tau)\, v_m ,\qquad (5)$$

where θ_m is the Hopscotch switch.

If we now define the Fourier solution of the *locally constant* equation

$$u_t + \beta\bar{u}\cdot u_x + \epsilon u_{xxx} = 0$$

($\bar{u}$ a locally constant value of u), as

$$u = \sum_{-\infty}^{\infty}{}_{(k)} C_k \exp i(kx - l(k)t),\qquad (6)$$

where k is the wavenumber and $l(k)$ is the frequency that is related by the frequency relation

$$l(k) = \beta\bar{u}k - \epsilon k^3,\qquad (7)$$

then the kth component of (6) may be written as

$$u_k = c_k \exp i(kx - l(k)t).\qquad (8)$$

If we increase t by an amount τ, then

$$\begin{aligned} u_k(t + \tau) &= C_k \exp i(kx - l(k)(t + \tau)) \\ &= \exp(-il(k)\tau)\cdot u_k(t). \end{aligned}\qquad (9)$$

Thus the kth component of the solution moves a distance $l(k)\tau$, in time τ, in a direction equal to the sign of $l(k)$ with preserved amplitude.

Consequently, if w_k is the kth component of the Fourier solution w of the difference approximation (4), then

$$w_k((m+1)\tau) = g(k)\, w_k(m\tau),\qquad (10)$$

where $g(k)$ is the amplification factor; see Richtmyer and Morton [12]. Thus, we write $g(k)$ as

$$g(k) = |\,g(k)|\exp(i\varphi),\qquad (11)$$

where $\varphi = \arg(g(k))$ is real.

Thus, the application of the difference operator L_h to u_k as defined by (8) produces a solution that has an amplitude modified by $|\,g(k)|$ and a phase modified by an amount φ in a direction opposite to the sign of φ.

If $\mid g(k)\mid = 1$ for all k, then the amplitude of the solution is not affected by the application of the difference operator. Such a scheme is called conservative, or nondissipative.

We can examine the phase properties of the scheme in a manner analogous to Greig and Morris [5]. We can summarize these as follows:

(i) $0 < \dfrac{-\arg g(k)}{l(k)\tau} < 1$

indicates a phase lag of the numerical solution relative to the theoretical solution;

(ii) $\dfrac{-\arg g(k)}{l(k)\tau} = 1$

gives exact phase; and

(iii) $\dfrac{-\arg g(k)}{l(k)\tau} > 1$

yields a phase gain of the numerical solution relative to the theoretical solution.

3. A Hopscotch Method for the KdV Equation

To describe the algorithm, we introduce the following notation. We discretize the space variable x into steps of size h and denote $x = ih$, $i = 0, 1,\ldots$.

Let $v \equiv v(ih, m\tau)$ be the difference solution at the gridpoints (i, m). Denote the usual finite-difference operators

$$\delta_x v_i{}^m \equiv v_{i+(1/2)}^m - v_{i-(1/2)}^m$$

$$H_x v_i{}^m \equiv v_{i+1}^m - v_{i-1}^m .$$

Also, for convenience, we let $f(u) = \frac{1}{2}u^2$ and denote $f_i^m = f(v_i^m)$. Further, we let $p = \tau/h$. Then for the KdV equation (1) we employ the simple divided difference operators for the differential operators in question in Eq. (1) so that Eq. (4) becomes

$$v_i^{m+1} = v_i{}^m - \frac{1}{2} p\beta H_x f_i{}^m - \frac{p\epsilon}{2h^2} H_x \delta_x{}^2 v_i{}^m, \tag{12}$$

so that L_h possesses the property of (but is not, in the original sense, equivalent to) an E-operator described in [3].

To describe the Hopscotch scheme based on this algorithm, we write the implicit version of (12), namely,

$$v_i^{m+1} = v_i^m - \frac{1}{2} p\beta H_x f_i^{m+1} - \frac{p\epsilon}{2h^2} H_x \delta_x^2 v_i^{m+1}. \tag{13}$$

Then, Eq. (5) becomes

$$v_i^{m+1} + \theta_i^{m+1} \left[\frac{1}{2} p\beta H_x f_i^{m+1} + \frac{p\epsilon}{2h^2} H_x \delta_x^2 v_i^{m+1} \right]$$

$$= v_i^m - \theta_i^m \left[\frac{1}{2} p\beta H_x f_i^m + \frac{p\epsilon}{2h^2} H_x \delta_x^2 v_i^m \right], \tag{14}$$

where

$$\theta_i^m = 1 \qquad i + m \qquad \text{even}$$
$$= 0 \qquad i + m \qquad \text{odd}.$$

In the notation of [3], this is an odd–even Hopscotch scheme.

If we write Eqs. (12) and (13) in terms of function values, then we have

$$v_i^{m+1} = v_i^m - \frac{1}{2} p\beta(f_{i+1}^m - f_{i-1}^m) - \frac{p\epsilon}{2h^2} (v_{i+2}^m - 2v_{i+1}^m + 2v_{i-1}^m - v_{i-2}^m) \tag{15}$$

and

$$v_i^{m+1} = v_i^m - \frac{1}{2} p\beta(f_{i+1}^{m+1} - f_{i-1}^{m+1}) - \frac{p\epsilon}{2h^2} (v_{i+2}^{m+1} - 2v_{i+1}^{m+1} + 2v_{i-1}^{m+1} - v_{i-2}^{m+1}). \tag{16}$$

Hence, to implement the scheme, we employ (15) for those gridpoints for which $(i + m)$ is even and (16) for those for which $(i + m)$ is odd.

To be precise, let us assume that m is even, so that Eq. (15) is applied for $i = 2, 4,..., N - 1$ where the solution is sought in the rectangle

$$(0 \leqslant x \leqslant Nh) \times (t > 0) \qquad \text{and} \qquad N \text{ is odd}.$$

Further, we can assume that we can determine the solution at $x = 0$, $-h$, Nh and $(N + 1)h$, either from given boundary conditions, or by an appropriate extrapolation technique (see Gourlay and Morris [4], for example.) In this paper, it will be sufficient to assume $v_0 = v_{-1} = v_N = v_{N+1} = 0$ for all t. The values obtained from Eq. (15) now are used in Eq. (16). Hence, rearranging (16), we have

$$v_i^{m+1} + \frac{p\epsilon}{2h^2} (v_{i+2}^{m+1} - v_{i-2}^{m+1}) = v_i^m - \frac{1}{2} p\beta(f_{i+1}^{m+1} - f_{i-1}^{m+1}) + \frac{p\epsilon}{h^2} (v_{i+1}^{m+1} - v_{i-1}^{m+1}), \tag{17}$$

where this algorithm, under present assumptions, is to be applied for

$i = 1, 3,..., N - 2$. All entries on the right-hand side of (17) are known, so that we can write this as

$$v_i^{m+1} + \frac{p\epsilon}{2h^2}(v_{i+2}^{m+1} - v_{i-2}^{m+1}) = K_i^m, \qquad i = 1, 3,..., N - 2, \tag{18}$$

where K_i^m is the right-hand side of Eq. (17). Hence, we can write the totality of equations as

$$A \cdot \mathbf{v}^{m+1} = \mathbf{K}, \tag{19}$$

where

$$A = \begin{bmatrix} 1 & \frac{p\epsilon}{2h^2} & & & \bigcirc \\ \frac{-p\epsilon}{2h^2} & 1 & \frac{p\epsilon}{2h^2} & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \frac{p\epsilon}{2h^2} \\ \bigcirc & & & \frac{-p\epsilon}{2h^2} & 1 \end{bmatrix} ; \qquad \mathbf{v}^{m+1} = \begin{bmatrix} v_1 \\ v_3 \\ \vdots \\ v_{N-2} \end{bmatrix}^{m+1}$$

and $\mathbf{K} = [K_1, K_3,..., K_{N-2}]$ where, in general, K_1 and K_{N-2} are suitably modified versions of the right-hand side of Eq. (17), taking into account $(-p\epsilon/(2h^2)) v_{-1}^{m+1}$ and $(p\epsilon/(2h^2)) v_N^{m+1}$, respectively.

For m odd, we will obtain the obvious change of subscripts in the vectors $\mathbf{v}$ and $\mathbf{K}$ in Eq. (19), but the coefficient matrix A will remain *unchanged*.

For N even, we will obtain a similar matrix of coefficients, however, at alternate time levels, i.e., values of m, the order of the matrix will vary by one as the number of unknowns corresponding to the implicit system (16) is either $\frac{1}{2}(N - 1)$ or $\frac{1}{2}(N - 2)$. Except for the change of order, the matrix coefficients are precisely those indicated and the algorithm employed to solve the tridiagonal system of equations is unchanged. Owing to the nature of the elements in the matrix A, it is easy to show that the temporary storage required to solve the system of equations comprises, at most, a one-dimensional array of $\frac{1}{2}(N - 1)$ elements. Consequently, the total storage comprises the array storing the $(N + 1)$ elements $\mathbf{v}^{m+1}$ and the temporary vector of $\frac{1}{2}(N - 1)$ elements. This compares with the Zabusky–Kruskal Scheme's storage requirement of two arrays each of $N + 1$ elements.

The form of the matrix A will play an important part in the efficient computation of the algorithm. We can produce a constant factorization of A that then requires only a back substitution at each time level to determine $\mathbf{v}^{m+1}$. The standard algorithm for computing the solution of a tridiagonal system of equations can be found in Mitchell [11].

4. ANALYSIS OF THE HOPSCOTCH METHOD

Let us write (16) for $m = m - 1$. Then

$$v_i^m = v_i^{m-1} - \frac{1}{2} p\beta(f_{i+1}^m - f_{i-1}^m) - \frac{p\epsilon}{2h^2} (v_{i+2}^m - 2v_{i+1}^m + 2v_{i-1}^m - v_{i-2}^m)$$

or

$$v_i^m = v_i^{m-1} - \frac{1}{2} p\beta(f_{i+1}^m - f_{i-1}^m) - \frac{p\epsilon}{2h^2} (v_{i+2}^m - v_{i-2}^m) + \frac{p\epsilon}{h^2} (v_{i+1}^m - v_{i-1}^m) \quad (20)$$

and

$$v_i^{m+1} = v_i^m - \frac{1}{2} p\beta(f_{i+1}^m - f_{i-1}^m) - \frac{p\epsilon}{2h^2} (v_{i+2}^m - v_{i-2}^m) + \frac{p\epsilon}{h^2} (v_{i+1}^m - v_{i-1}^m). \quad (21)$$

In operator notation, we then have

$$\left(1 + \frac{p\epsilon}{2h^2} H_{2x}\right) v_i^m = v_i^{m-1} - \frac{1}{2} p\beta H_x f_i^m + \frac{p\epsilon}{h^2} H_x v_i^m \quad (22)$$

and

$$v_i^{m+1} = \left(1 - \frac{p\epsilon}{2h^2} H_{2x}\right) v_i^m - \frac{1}{2} p\beta H_x f_i^m + \frac{p\epsilon}{h^2} H_x v_i^m. \quad (23)$$

We eliminate v_i^m from (22) and (23), multiply (22) by $(1 - (p\epsilon/(2h^2)) H_{2x}$ and (23) by $(1 + (p\epsilon/(2h^2)) H_{2x})$, and add. The resulting eqaution is

$$\left(1 + \frac{p\epsilon}{2h^2} H_{2x}\right) v_i^{m+1} = \left(1 - \frac{p\epsilon}{2h^2} H_{2x}\right) v_i^{m-1} - p\beta H_x f_i^m + \frac{2p\epsilon}{h^2} H_x v_i^m. \quad (24)$$

Hence, the odd-even Hopscotch algorithm is equivalent to the three-level scheme (24). Note that this is not equivalent to the Zabusky–Kruskal scheme

$$v_i^{m+1} = v_i^{m-1} - p\beta H_x f_i^m - (p\epsilon/h^2) H_x \delta_x^2 v_i^m. \quad (25)$$

It can be shown that the method has a truncation error which is $\tau O((\tau/h)^2 + h^2 + \tau^2)$.

We may examine the von Neumann stability of the scheme in the manner explained in Section 2. The amplification factor $g(k)$ satisfies

$$(1 + 2\lambda i \sin 2\xi) g^2 + 2i \sin \xi(p\beta u - 4\lambda) g - (1 - 2\lambda i \sin 2\xi) = 0, \quad (26)$$

where $\xi = kh$, with k the wavenumber, h the mesh spacing in the space direction, and $\lambda = p\epsilon/(2h^2)$.

To analyze conditions under which $|g| \leqslant 1$, we use Miller's analysis [10]. Define

$$f(z) = (1 + 2\lambda i \sin 2\xi) z^2 + 2i \sin \xi \cdot (p\beta u - 4\lambda)z - (1 - 2\lambda i \sin 2\xi)$$

and suppose $z^* = 1/\bar{z}$, i.e.,

$$f(z^*) = (1 + 2\lambda i \sin 2\xi) \bar{z}^{-2} + 2i \sin \xi \cdot (p\beta u - 4\lambda) \bar{z}^{-1} - (1 - 2\lambda i \sin 2\xi),$$

and

$$\overline{f(z^*)} = (1 - 2i\lambda \sin 2\xi) z^{-2} - 2i \sin \xi \cdot (p\beta u - 4\lambda) z^{-1} - (1 + 2i\lambda \sin 2\xi).$$

Therefore, with $f^*(z) = z^2 \overline{f(z^*)}$,

$$f^*(z) = (1 - 2i\lambda \sin 2\xi) - 2i \sin \xi \cdot (p\beta u - 4\lambda)z - (1 + 2i\lambda \sin 2\xi) z^2.$$

Then define the Bezout resultant:

$$\check{f} = (f^*(0) f(z) - f(0) f^*(z))/z.$$

Hence, if $f^*(0) = 1 - 2i\lambda \sin 2\xi$, and $f(0) = -(1 - 2i\lambda \sin 2\xi)$, then clearly $|f^*(0)| \geqslant |f(0)|$.

For f to be von Neumann, we must show that

(i) $\check{f} \equiv 0$, and

(ii) f' is von Neumann.

Now,

$$\check{f} = z^{-1}[(1 - 2i\lambda \sin 2\xi)\{(1 + 2i\lambda \sin 2\xi) z^2 + 2i \sin \xi \cdot (p\beta u - 4\lambda)z$$
$$- (1 - 2i\lambda \sin 2\xi)\} + (1 - 2i\lambda \sin 2\xi)\{(1 - 2i\lambda \sin 2\xi) - 2i \sin \xi \cdot (p\beta u - 4\lambda)z$$
$$- (1 + 2\lambda i \sin 2\xi) z^2\}] \equiv 0.$$

Then, for f to be von Neumann, we are required to show (ii) above:

$$f'(z) = 2(1 + 2i\lambda \sin 2\xi)z + 2i \sin \xi \cdot (p\beta u - 4\lambda).$$

Hence, we require that $|z| \leqslant 1$ for this function. That is, we require

$$|z| = \left| \frac{-2i \sin \xi \cdot (p\beta u - 4\lambda)}{2(1 + 2i\lambda \sin 2\xi)} \right| \leqslant 1 \qquad \forall \xi. \qquad (27)$$

Note that by ensuring $|z| \leqslant 1$, the resulting zeroes of $f(z)$ lie on the unit circle [10]. Therefore, the scheme is conservative.

Now

$$| z |^2 = \frac{4 \sin^2 \xi \cdot (p\beta u - 4\lambda)^2}{4(1 + 4\lambda^2 \sin^2 2\xi)} .$$

Hence, by (27), we require

$$4 \sin^2 \xi \cdot (p\beta u - 4\lambda)^2 \leqslant 4(1 + 4\lambda^2 \sin^2 2\xi)$$

that is

$$\sin^2 \xi \cdot (p\beta u - 4\lambda)^2 \leqslant 1 + 4\lambda^2 \sin^2 2\xi.$$

Now $\lambda = (p\epsilon/(2h^2))$, and so

$$\sin^2 \xi \cdot \left(p\beta u - \frac{2p\epsilon}{h^2}\right)^2 \leqslant 1 + \frac{p^2\epsilon^2}{h^4} \sin^2 2\xi$$

for stability. That is:

$$p^2 \left\{\sin^2 \xi \left(\beta^2 u^2 - \frac{4\epsilon\beta u}{h^2} + \frac{4\epsilon^2}{h^4}\right) - \frac{4\epsilon^2}{h^4} \sin^2 \xi(1 - \sin^2 \xi)\right\} \leqslant 1,$$

$$p^2 \left\{\sin^2 \xi \cdot \left(\beta^2 u^2 - \frac{4\epsilon\beta u}{h^2}\right) + \frac{4\epsilon^2}{h^4} \sin^4 \xi\right\} \leqslant 1, \qquad (28)$$

$$p^2 \sin^2 \xi \cdot \left[\beta^2 u^2 - \frac{4\epsilon\beta u}{h^2} + \frac{4\epsilon^2}{h^4} \sin^2 \xi\right] \leqslant 1.$$

The maximum value of the left-hand side of (28) occurs when $\sin^2 \xi = 1$. Hence,

$$p^2 \left(\beta^2 u^2 - \frac{4\epsilon\beta u}{h^2} + \frac{4\epsilon^2}{h^4}\right) \leqslant 1,$$

i.e.,

$$p^2 \left(\beta u - \frac{2\epsilon}{h^2}\right)^2 \leqslant 1.$$

Hence, the Hopscotch method for the KdV equation is conservative and is stable if

$$p \mid \beta u - (2\epsilon/h^2)\mid \leqslant 1. \qquad (29)$$

This condition, for the values of the parameters of interest here, is considerably less stringent than the stability condition for the Zabusky–Kruskal method, namely,

$$p(\beta \mid u \mid + (4\epsilon/h^2)) \leqslant 1. \qquad (30)$$

If we return to Eq. (26), we know from the above analysis that the zeros lie on the unit circle and so we have

$$g(k) = \exp(i\varphi).$$

Substitution of this expression into Eq. (26) gives us a system to solve for φ, namely,

$$4(1 + A^2) s^2 + 4Bs + B^2 - 4A^2 = 0$$
$$4(1 + A^2) c^2 + 4ABc + (B^2 - 4) = 0, \tag{31}$$

where $s = \sin \varphi$, $c = \cos \varphi$, $A = 2\lambda \sin 2\xi$, $B = 2 \sin \xi(p\beta u - 4\lambda)$, $\lambda = (p\epsilon/2h^2)$, and $\xi = kh$, with k the wavenumber.

Clearly, solution of this system produces two roots, s_1, s_2 and c_1, c_2, for each of the constituent equations. However, the requirement $s_i^2 + c_j^2 = 1$ reduces the number of possibilities to two combinations, as expected by a solution of Eq. (26).

Thus, we obtain two roots, φ_1 and φ_2, by taking the arctangent of the ratio of the appropriate roots of Eq. (31). Consequently, we may write

$$g(k) = \alpha_k \exp(i\varphi_1) + \beta_k \exp(i\varphi_2),$$

where α_k and β_k are parameters. To determine α_k and β_k, we require the numerical solution given by

$$w(m\tau) = \sum_{-\infty}^{\infty}{}_{(k)} c_k[\alpha_k \exp(im\varphi_1) + \beta_k \exp(im\varphi_2)] \exp(ikx) \tag{32}$$

to reproduce the initial condition given by Eq. (6) with $t = 0$, namely,

$$u_k = \sum_{-\infty}^{\infty}{}_{(k)} c_k \exp(ikx),$$

i.e., $\alpha_k + \beta_k = 1$ so that $\beta_k = 1 - \alpha_k$.

To obtain α_k, we equate Eq. (32) with $m = 1$ to the Hopscotch solution given by Eq. (13) with $m = 0$, namely,

$$\alpha_k \exp(i\varphi_1) + (1 - \alpha_k) \exp(i\varphi_2) = (1 + ip \sin \xi[\beta\bar{u} - (2\epsilon/h^2)(1 - \cos \xi)])^{-1}.$$

The numerical solution of this set of equations, for a sequence of values of h and τ and for the first 15 wavenumbers, shows that $\alpha_k = 1$ and $\beta_k = 0$. Hence, we may conclude that the contribution made by $\beta_k \exp(i\varphi_2)$ is zero and we may write $g(k) = \exp(i\varphi_1)$.

For the values of h and τ used in our experiments, we have tabulated $l(k)$ against $-\varphi_1/\tau$. The variable $l(k)$ is the distance moved by the theoretical solution in unit

TABLE I

k	l(k)	$-\varphi_1/\tau$ h = 0.05 $\tau = 0.025$	$-\varphi_1/\tau$ h = 0.01 $\tau = 0.0005$
1	0.999	0.999	0.999
2	1.996	1.993	1.996
3	2.986	2.975	2.986
4	3.969	3.941	3.968
5	4.939	4.886	4.937
6	5.895	5.806	5.893
7	6.833	6.693	6.828
8	7.752	7.546	7.743
9	8.647	8.359	8.635
10	9.516	9.129	9.500
11	10.355	9.851	10.334
12	11.163	10.526	11.136
13	11.936	11.148	11.902
14	12.671	11.715	12.630
15	13.366	12.227	13.316

time and $-\varphi_1/\tau$ is the distance moved by the computed solution. These data are given in Table I.

The comparison of $l(k)$ and $-\varphi_1/\tau$ also was carried out for larger values of h and τ. It was found, as would be expected, that the difference increased as h and τ increased. The results quoted in Table I correspond to the values of h and τ used in the numerical experiments in Section 5.

5. NUMERICAL RESULTS

We computed solutions to the KdV equation (1) subject to the following conditions:

(a) The initial condition

$$u(x, 0) = 3c \operatorname{sech}^2 (Ax + D) \tag{33}$$

and $u(0, t) = u(2, t) = 0$, for all t.

(b) The initial condition

$$u(x, 0) = 3c_1 \operatorname{sech}^2 (A_1x + D_1) + 3c_2 \operatorname{sech}^2 (A_2x + D_2) \tag{34}$$

and $u(0, t) = u(2, t) = 0$, for all t.

Equations (1) and (33) have the theoretical solution

$$u(x, t) = 3c \, \text{sech}^2 \, (Ax - Bt + D) \tag{35}$$

provided $A = \frac{1}{2}(\beta c/\epsilon)^{1/2}$ and $B = \frac{1}{2}\beta c(\beta c/\epsilon)^{1/2}$. Eq. (35) represents a single soliton moving in the direction of increasing x.

For the experiment with initial condition (33), the values of the parameters were as follows: $c = 0.3$, $D = -6$, $\beta = 1$, $\epsilon = 4.84_{10} - 4$. The experiment was carried out on an initial interval of $[0, 2]$ for a sequence of values of h and τ. A comparison was made, with the theoretical solution given by (35) and with the Zabusky–Kruskal method (25), for the same values of parameters.

In Table II, we have tabulated the L_2 and L_∞ errors of both the Hopscotch and Zabusky–Kruskal methods for various h and τ.

TABLE II

		Zabusky–Kruskal		Hopscotch	
	T	L_2	L_∞	L_2	L_∞
$h = 0.05, \tau = 0.025$	0.25	0.03464	0.0194	0.06121	0.0327
	0.50	0.12268	0.0635	0.12241	0.0674
	0.75	0.21044	0.1224	0.18135	0.0993
	1.00	0.29819	0.1614	0.22810	0.1416
		$\times_{10} + 3$	$\times_{10}3$	$\times_{10}3$	$\times_{10}3$
$h = 0.01, \tau = 0.0005$	0.25	5.945	2.049	3.788	1.113
	0.50	13.173	4.225	9.277	2.136
	0.75	21.079	6.364	14.138	3.542
	1.00	28.661	8.133	18.725	4.906

The graphs in Figs. 1 and 2 show the solution at time $T = 0.25$ and $T = 1$. The actual solution given by (35) also is drawn on the figures, but the graphs cannot be distinguished due to the closeness of the numerical solution to the theoretical one.

We were required, in addition, to investigate the behavior of the Hopscotch method for a problem representing the interaction of two solitons. The initial condition (34) represents two solitons, one with amplitude c_1 placed initially at $x = -D_1/A_1$ and the second with amplitude c_2 placed at $x = -D_2/A_2$. As is well
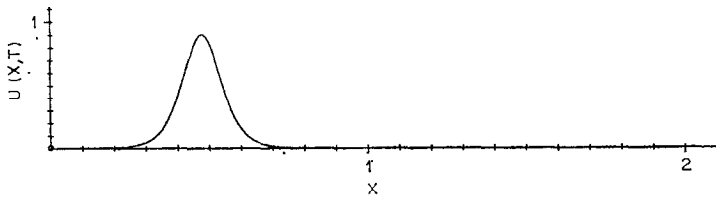
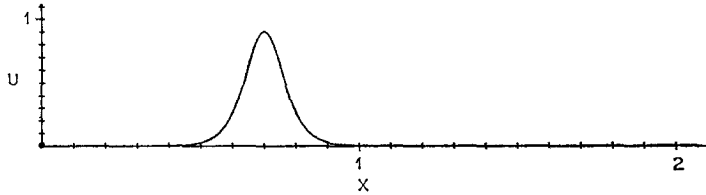FIG. 1. Numerical and theoretical solution of (1) and (30) at time $T = 0.25$.

FIG. 2. Numerical and theoretical solution of (1) and (30) at time $T = 1$.

known, a soliton with larger amplitude has a greater velocity than another soliton with smaller amplitude. Consequently, choosing $c_1 > c_2$ and $-D_1/A_1 < -D_2/A_2$ should ensure an interaction of the two waves with increasing time. However, we cannot compare our numerical results in this case with a theoretical solution since despite the fact that both constituents in themselves are solutions of the KdV equation, the combination suggested by the initial condition (34) is not, as the principle of superposition does not hold.

The interaction experiment with initial condition (34) was run with the following values of parameters: $c_1 = 0.3$, $c_2 = 0.1$, and $D_1 = D_2 = -6$. In this case, the experiment was run from $T = 0$ to $T = 3$ to allow the interaction to take place. This represented some 4800 cycles of the method.

The initial condition (34) is represented in Fig. 3. This shows two wave pulses,
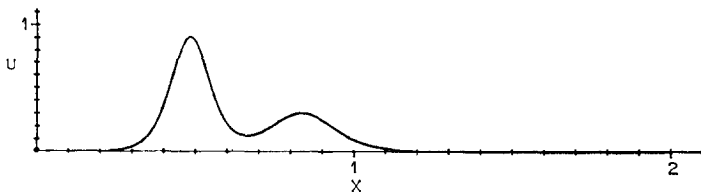

FIG. 3. Initial condition (31).

with the larger on the left. As time increases, the larger soliton (which is faster moving) catches up with the smaller until, at time $T = 0.75$, the smaller pulse is in the process of being absorbed, having lost its solitary wave identity (see Fig. 4).
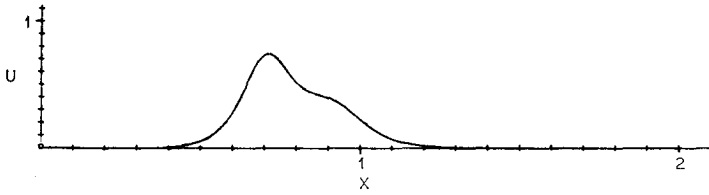
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FIG. 4. Numerical solution of (1) and (31) at time $T = 0.75$.

The overlapping process continues until, by time $T = 1.5$, the larger pulse has overtaken the smaller one and is in the process of separating, as in Fig. 5.
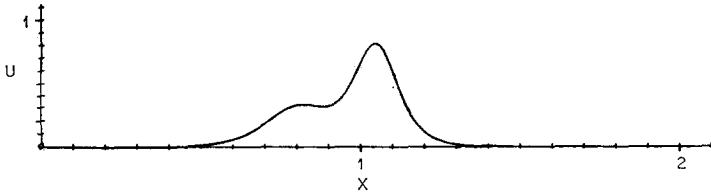

FIG. 5. Numerical solution of (1) and (31) at time $T = 1.5$.

By time $T = 3.0$, the interaction is complete and the larger soliton has separated completely from the smaller one. This is seen in Fig. 6. The Zabusky–Kruskal
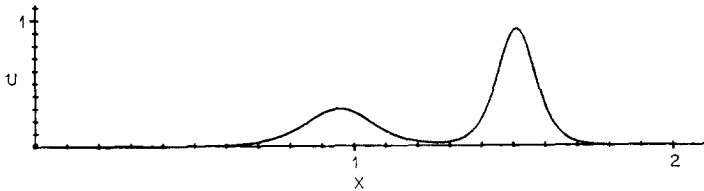

FIG. 6. Numerical solution of (1) and (31) at time $T = 3$.

scheme (25) was run for the same values of h and τ for the present initial condition and, as expected from the stability condition (30), the method became unstable before the interaction took place. We have not run the present schemes for smaller τ to coincide with the stability condition for the Zabusky–Kruskal scheme, since this would require an unwarranted number of time steps to produce the observed interaction. However, for smaller τ, we would expect the two methods to produce similar results.

6. Conclusions

A Hopscotch algorithm has been proposed for the numerical solution of the KdV equation. The analysis of the method indicates that the algorithm has desirable properties, namely, the method is conservative and has small phase error.

The numerical solution of the KdV equation, subject to a single soliton solution whose theoretical solution can be determined, provides results indistinguishable, on the scale used, from the theoretical solution. Further, the algorithm exhibits its ability to compute the interaction of solitary waves previously described in Zabusky and Kruskal [19]. We have verified that two solitary waves may coalesce for a brief period and then separate again with their original profiles intact, but with their positions interchanged. The Hopscotch algorithm produced solitons subsequent to the interaction that possessed amplitudes altered from the original solitons by only 1 %, which is consistent with that reported in [19].

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